

### Abstract

We construct a perturbation theory for the  $SU(2)$  non-linear  $\sigma$ -model in  $2 + 1$  dimensions using a polynomial, first-order formulation, where the variables are a non-Abelian vector field  $L_\mu$  (the left  $SU(2)$  current), and a non-Abelian pseudovector field  $\theta_\mu$ , which imposes the condition  $F_{\mu\nu}(L) = 0$ . The coordinates on the group do not appear in the Feynman rules, but their scattering amplitudes are easily related to those of the currents. We show that all the infinities affecting physical amplitudes at one-loop order can be cured by normal ordering, presenting the calculation of the full propagator as an example of an application.

Key words: sigma-model, renormalization.

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# The $SU(2)$ Non-Linear $\sigma$ -Model in $2 + 1$ Dimensions: Perturbation Theory in a Polynomial Formulation

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# 1 Introduction.

The non-linear  $\sigma$ -model [1] has proven to be a very fruitful subject of research, showing remarkable properties both from the perturbative and non-perturbative points of view. In non-perturbative studies, the presence of interesting objects (instantons, topological charges, Skyrmions) is a consequence of the fact that the Nambu-Goldstone fields provide a coordinatization of a (Riemmanian) manifold  $M$ , which has a non-trivial geometry. Those mappings can in some cases be partitioned into different homotopy classes. The unusual perturbative properties [2] are also a consequence of the non-triviality of  $M$ : the invariant metric on the manifold is field-dependent, and thus the Lagrangian becomes non-polynomial in the coordinate fields.

Despite this complication, the renormalizability of the  $1+1$  model has been established some time ago [2]. In higher dimensions, the model is non-renormalizable under the usual perturbative expansion. However, it was shown [3] that in  $2+1$  dimensions the  $O(N)$  model is renormalizable when expanded in  $1/N$ . This expansion is essentially different to the usual loopwise perturbative expansion. Indeed, a given order in the former carries a non-analytic dependence in the perturbative parameter of the latter. Although the perturbative properties of the model are substantially improved by this  $1/N$  expansion, the non-perturbative ones may be changed, since the topology of  $M$ , and hence the classification into homotopy classes, depends on  $N$ .

In this paper we will study the  $SU(2)$  non-linear  $\sigma$ -model in  $2+1$  dimensions, using a loopwise expansion *in terms of a different set of variables*. These variables are the left currents associated with one of the global  $SU(2)$  symmetries. The idea that they provide a better basis of fields to study (regulated) Chiral Perturbation Theory in  $3+1$  dimensions was introduced by Slavnov [4], who showed that the counterterms could be written in terms of  $L_\mu$  only. The actual calculations, however, required the

introduction of the coordinate fields in the usual non-polynomial Lagrangian during the intermediate steps. A completely coordinate-independent and polynomial Lagrangian was later introduced [5, 6]; we will follow that approach here. As the basic fields are non-linearly related to the coordinates on the group (‘pions’), some renormalization aspects will change. Indeed, as the currents transform in a simpler way under the group operations, the counterterms should be better organized when written in terms of them. Of course, the geometric properties will not change, since the manifold  $M$  is unchanged. Indeed, some of them appear more explicitly in this formulation.

We will show that at one-loop level<sup>1</sup> all the diagrams contributing to physical amplitudes are finite, despite the suggestion to the contrary implied by naive power-counting. Indeed, the only necessary counterterms are the ones due to normal-ordering.

We explicitly calculate the full one-loop propagator for the left current, which can be related to the pion propagator when evaluated on-shell. The usual non-polynomial Lagrangian for this model is

$$\mathcal{L} = \frac{1}{2} g \operatorname{tr}(\partial_\mu U^\dagger \partial^\mu U) , \quad (1)$$

where  $U(x)$  belongs to  $SU(2)$  and  $g$  is a constant with dimensions of mass. We use the spacetime metric  $\eta_{\mu\nu} = \operatorname{diag}(1, -1, -1)$ . The group elements  $U(x)$  can be parametrized in terms of the ‘pion’ fields  $\pi^a(x)$ ,  $a = 1, 2, 3$ , coordinates in the Lie algebra of  $SU(2)$

$$U(x) = \exp[\pi(x)] \quad , \quad \pi(x) = \pi^a(x) \lambda^a . \quad (2)$$

The generators  $\lambda^a$  satisfy

$$[\lambda^a, \lambda^b] = \epsilon^{abc} \lambda^c , \quad \lambda^{a\dagger} = -\lambda^a , \quad \operatorname{tr}(\lambda^a \lambda^b) = -\delta_{ab} , \quad (3)$$

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<sup>1</sup>We will not deal with the problem of renormalizability to all orders.

where  $\epsilon_{abc}$  is the Levi-Civita symbol.

In refs. [5, 6] a polynomial representation of the non-linear  $\sigma$ -model was introduced; let us briefly explain it for the particular case of the  $SU(2)$  model in 2+1 dimensions. It is constructed in terms of a non-Abelian ( $SU(2)$ ) vector field  $L_\mu$  plus a non-Abelian antisymmetric tensor field  $\theta_{\mu\nu}$ , with the Lagrangian

$$\mathcal{L} = \frac{1}{2}g^2 L_\mu \cdot L^\mu + g \theta_{\mu\nu} \cdot F^{\mu\nu}(L) \quad (4)$$

where the fields  $L_\mu$  and  $\theta_{\mu\nu}$  are defined by their components in the basis of generators of the adjoint representation of the Lie algebra of  $SU(N)$ ; i.e.,  $L_\mu(x)$  is a vector with components  $L_\mu^a$ ,  $a = 1, 2, 3$ , and analogously for  $\theta_{\mu\nu}$ . The components of  $F_{\mu\nu}$  in the same basis are:  $F_{\mu\nu}^a(L) = \partial_\mu L_\nu^a - \partial_\nu L_\mu^a + g^{\frac{1}{2}}\epsilon^{abc}L_\mu^b L_\nu^c$ . The dots mean  $SU(2)$  scalar product, for example:  $L_\mu \cdot L^\mu = \sum_{a=1}^3 L_\mu^a L_a^\mu$ . We will also use the ‘cross product’  $A \times B$ , to mean  $(A \times B)^a = \epsilon^{abc}A^b B^c$ . The exponents in the factors of  $g$  are chosen in order to make the fields have the appropriate canonical dimension in 2 + 1 dimensions.

The Lagrange multiplier  $\theta_{\mu\nu}$  imposes the constraint  $F_{\mu\nu}(L) = 0$ , which is equivalent [7] to  $L_\mu = g^{-\frac{1}{2}}U\partial_\mu U^\dagger$ , where  $U$  is an element of  $SU(2)$ . When this is substituted back in (4), (1) is obtained. This polynomial formulation could be thought of as a concrete Lagrangian realization of the Sugawara theory of currents [8], where all the dynamics is defined by the currents, the energy-momentum tensor, and their algebra. Indeed,  $L_\mu$  corresponds to one of the conserved currents of the non-polynomial formulation, due to the invariance of  $\mathcal{L}$  under global (left)  $SU(N)$  transformations of  $U(x)$ . The energy-momentum tensor following from (4) is indeed a function of  $L_\mu$  only:

$$T^{\mu\nu} = g^2(L^\mu \cdot L^\nu - \frac{1}{2}g^{\mu\nu}L^2), \quad (5)$$

as can be verified by rewriting (4) in a generally covariant form and taking the functional derivative with respect to the spacetime metric. To avoid working with

the 2-index tensor field  $\theta_{\mu\nu}$  we write it in terms of its dual, which in  $2+1$  becomes a pseudovector (we assume  $\pi$  to be a scalar, but everything can be easily translated to the case of a pseudoscalar field)  $\theta_\mu^a = \frac{1}{2}\epsilon_{\mu\nu\lambda}\theta^{a\nu\lambda}$ . With this convention, eq.(4) becomes

$$\mathcal{L} = \frac{1}{2}g^2 L_\mu \cdot L^\mu + \frac{1}{2}g \epsilon^{\mu\nu\lambda} \theta_\mu \cdot F_{\nu\lambda}(L) . \quad (6)$$

The action corresponding to (6) is invariant under the gauge transformations

$$\theta_\mu(x) \rightarrow \theta_\mu(x) + D_\mu \omega(x) , \quad (7)$$

where  $D$  is the covariant derivative with respect to  $L$ , defined by:  $D_\mu \omega = \partial_\mu \omega + g^{\frac{1}{2}} L_\mu \times \omega$ . The variation of the action vanishes as a consequence of the Bianchi identity for  $L_\mu$ :

$$\epsilon^{\mu\nu\rho} D_\mu F_{\nu\rho}(L) = 0 . \quad (8)$$

This kind of symmetry has been found in many different contexts; for example, when considering the dynamics of a two-form gauge field [9]. The presence of this symmetry here can be understood as follows: As  $F_{\mu\nu}$  satisfies the Bianchi identity (8), the system of equations  $F_{\mu\nu} = 0$  has some redundancy. Hence, it should be possible to find an equivalent system with a smaller number of equations (if one sacrifices locality). Thus one really needs a smaller number of components in the Lagrange multiplier to impose the constraint, and some of them should be redundant. This is what the symmetry (7) says. Consequences of this symmetry in the canonical version of the model were considered in ref. [10].

Although superficially equal to the infinitesimal version of a non-Abelian  $SU(2)$  gauge transformation, (7) is essentially different:  $\theta_\mu$  transforms with the covariant derivative with respect to  $L_\mu$ , and  $L_\mu$  itself does not transform. Hence, the gauge group is *Abelian*, thus finite and infinitesimal transformations have the same form.

The structure of the paper is as follows: In section 2 we discuss some properties of the gauge-fixed version of (4) and derive the Feynman rules. In section 3 we

present the calculation of the full propagators for the fields  $L$  and  $\theta$  to one loop order, and in section 4 we discuss the one-loop finiteness of the physical amplitudes. Section 5 contains our conclusions. Appendix A deals with the proper definition of the measure for the integration over  $L_\mu$ , and Appendix B is dedicated to clarifying some non-perturbative aspects of the ghost Lagrangian.

## 2 The model

### 2.1 Gauge-fixing and $BRST$ symmetry.

From the discussion presented in the Introduction, the model we consider is defined by the Lagrangian

$$\mathcal{L}_{inv} = \frac{1}{2}g^2 L_\mu \cdot L^\mu + \frac{1}{2}g \theta_\mu \cdot \epsilon^{\mu\nu\lambda} F_{\nu\lambda}(L) . \quad (9)$$

The suffix ‘*inv*’ for  $\mathcal{L}$  in eq.(9) means that it is gauge-invariant under the transformations (7). Using the standard Faddeev-Popov technique, and a covariant gauge-fixing, we get the full Lagrangian

$$\mathcal{L} = \mathcal{L}_{inv} + \mathcal{L}_{g.f.} + \mathcal{L}_{gh.} , \quad (10)$$

where  $\mathcal{L}_{g.f.}$  and  $\mathcal{L}_{gh.}$  are the gauge-fixing and ghost Lagrangians, respectively. They are given by

$$\mathcal{L}_{g.f.} = \frac{b^2}{2\lambda} + b \cdot \partial_\mu \theta^\mu , \quad \mathcal{L}_{gh} = -i\partial_\mu \bar{c} \cdot D^\mu c , \quad (11)$$

where  $\mathcal{L}_{g.f.}$  was rewritten in the Nakanishi-Lautrup form, in order to exhibit the nilpotence of the  $BRST$ -transformations (see below). So far the gauge-fixing and ghost Lagrangians look very much like the corresponding ones of the Yang-Mills theory in  $2+1$  dimensions. However, there is an important difference: the gauge transformations we are dealing with here affect  $\theta_\mu$  rather than  $L_\mu$ , but the ghost Lagrangian is still defined using the covariant derivative *with respect to*  $L_\mu$ <sup>2</sup>. The

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<sup>2</sup>Some issues about the Faddeev-Popov Lagrangian are further discussed in Appendix B.

corresponding *BRST* transformations are thus different to the ones of Yang-Mills theory:

$$\begin{aligned} sL_\mu &= 0 & s\theta_\mu &= D_\mu c \\ s\bar{c} &= ib & sc &= 0, \end{aligned} \tag{12}$$

where  $s$  is a nilpotent fermionic operator ( $s^2 = 0$ ). One easily verifies the invariance of the action under (12), since  $\mathcal{L}$  changes by a total derivative

$$s\mathcal{L} = \partial_\mu (b \cdot D^{\mu c} + \frac{1}{2}gc \cdot \epsilon^{\mu\nu\lambda} F_{\nu\lambda}) . \tag{13}$$

The conserved Noether current which follows from this global symmetry is

$$J_B^\mu = b \cdot D^\mu c - \frac{1}{2}gc \cdot \epsilon^{\mu\nu\rho} F_{\nu\rho} , \tag{14}$$

and its associated *BRST* charge becomes:

$$Q_B = \int d^2x (b \cdot D^0 c - gc \cdot \epsilon_{jk} \partial_j L_k) . \tag{15}$$

The equations of motion for the gauge-fixed Lagrangian (10) can be written as

$$\begin{aligned} \partial_\mu L^\mu &= s(\mathcal{F}) \\ F_{\mu\nu}(L) &= s(\mathcal{G}_{\mu\nu}) \end{aligned} \tag{16}$$

where  $\mathcal{F} = ig^{-\frac{3}{2}}\partial_\mu \bar{c} \times \theta^\mu$ , and  $\mathcal{G}_{\mu\nu} = -i\epsilon_{\mu\nu\rho}\partial^\rho \bar{c}$ . The ones for (9), after eliminating  $\theta$ , are equal to (16) except for the fact that their right hand sides are equal to zero. As the rhs in (16) are *BRST*-variations of functions, they can be written as the anticommutator of the *BRST*-charge (15) with the corresponding functions, and so we conclude that when the equations are sandwiched between *BRST*-invariant states, their rhs are zero. Thus they coincide with the equations of motion which follow from the gauge-invariant Lagrangian, and the physical dynamics is gauge-independent.



## 2.2 Perturbation theory and Feynman rules.

Wick-rotating and integrating Lagrangian (10) over  $b$ , we get the Euclidean Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}g^2 L_\mu \cdot L_\mu + \frac{1}{2} i g \epsilon_{\mu\nu\rho} \theta_\mu \cdot F_{\nu\rho} \\ &+ \frac{\lambda}{2} (\partial_\mu \theta_\mu)^2 - i \bar{c} \partial \cdot (Dc) ,\end{aligned}\tag{17}$$

which can be conveniently split into free and interaction parts:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{int} \\ \mathcal{L}_0 &= \frac{1}{2} g^2 L_\mu \cdot L_\mu + i g \epsilon_{\mu\nu\rho} \theta_\mu \cdot \partial_\nu L_\rho + \frac{\lambda}{2} (\partial_\mu \theta_\mu)^2 - i \bar{c} \cdot \partial^2 c \\ \mathcal{L}_{int} &= \frac{i}{2} g^{\frac{3}{2}} \epsilon_{\mu\nu\rho} \theta_\mu \cdot L_\nu \times L_\rho - i g^{\frac{1}{2}} \bar{c} \cdot \partial_\mu (L_\mu \times c) .\end{aligned}\tag{18}$$

The free Lagrangian  $\mathcal{L}_0$  determines the free propagators, and the interaction one  $\mathcal{L}_{int}$  the vertices, as usual. Note that there is a *quadratic* mixing between  $\theta_\mu$  and  $L_\mu$ , thus there will also be a mixed free propagator for those fields. The propagators and vertices in momentum space are represented graphically in Fig.1. The analytic expressions for the free propagators are:

$$\begin{aligned}\langle L_\mu L_\nu \rangle &= \frac{1}{g^2} \frac{k_\mu k_\nu}{k^2} \\ \langle \theta_\mu \theta_\nu \rangle &= \frac{\delta_{\mu\nu}}{k^2} - \frac{\lambda - 1}{\lambda} \frac{k_\mu k_\nu}{(k^2)^2} \\ \langle L_\mu \theta_\nu \rangle &= -\frac{1}{g} \epsilon_{\mu\rho\nu} \frac{k_\rho}{k^2} \\ \langle c \bar{c} \rangle &= \frac{-i}{k^2} .\end{aligned}\tag{19}$$

Due to the presence of a propagator mixing  $L$  and  $\theta$ , these fields could be regarded as two different components (in some ‘internal space’) of a single vector field  $\Phi_\mu$ . With this convention all the propagators involving  $L$  and  $\theta$  are particular matrix elements of the propagator for  $\Phi$ . We define the two components of  $\Phi$  by:  $\Phi^L = L$

and  $\Phi^\theta = \theta$ . We will however keep the old notation whenever it is more useful, for example to analyze large-momentum behaviours.

Note that there are two different vertices which follow from  $\mathcal{L}_{int}$ , one involves two ghost lines and one of  $L_\mu$ , and is identical to the equivalent one in Yang-Mills theory (with  $L_\mu$  as the gauge field); the other involves two lines of  $L_\mu$  and one of  $\theta_\mu$ .

In Euclidean three-dimensional spacetime, identifying all the points at infinity, the configurations can be classified by their winding numbers  $n$ , given by

$$\begin{aligned} n &= \frac{g^{3/2}}{12\pi^2} \int d^3x \epsilon_{\mu\nu\lambda} \epsilon^{abc} L_\mu^a L_\nu^b L_\lambda^c \\ &= \frac{1}{2\pi^2} \int d^3x \det(g^{1/2} L_\mu^a) . \end{aligned} \quad (20)$$

where  $L_\mu^a$  is regarded as a  $3 \times 3$  matrix in the indices  $a$  and  $\mu$ . The  $\Theta$  vacua term can then be introduced by adding to the action the following piece

$$S_\Theta = i \Theta n . \quad (21)$$

Such a term can also be justified as coming from the integration of very massive fermions minimally coupled to  $L$ , since they generate a Chern-Simons term which, when  $F_{\mu\nu} = 0$ , can be rewritten in terms of  $n$  only. We do not consider the effect of this term here, but just mention that it can be incorporated without spoiling the gauge symmetry. It generates an extra local vertex with three  $L$ 's.

### 2.3 Loop expansion and its meaning.

Let us briefly explain the meaning of the perturbative expansion in this model. We should first point out that a loop expansion cannot be understood *apriori* as a semi-classical expansion, since the functional integral has a weight which is not exactly of the form  $\exp(-\frac{S}{\hbar})$ , because *there is no factor of  $\hbar$  in the Lagrange-multiplier term*. We would like to relate here an expansion in the number of loops to the usual one of the non-polynomial formulation (which is a semiclassical approximation, but can

also be regarded as a low-momentum expansion, as in Chiral Perturbation Theory). We calculate the factor of  $g$  we have in front of a given proper diagram  $G$  in the polynomial formulation. We note that this factor (denoted  $f(G)$ ) can be expressed as:

$$f(G) = g^K, \quad K = \frac{3}{2}V_a + \frac{1}{2}V_b - (2I_{LL} + I_{L\theta}), \quad (22)$$

where:  $V_a$  = number of  $LL\theta$  vertices;  $V_b$  = number of  $\bar{c}cL$  vertices;  $I_{LL}$  = number of internal  $L$ - $L$  propagators, and  $I_{L\theta}$  = number of internal  $L$ - $\theta$  propagators. By some simple transformations,  $K$  can be rewritten as

$$K = E_L - \frac{1}{2}(V_a + V_b), \quad (23)$$

with  $E_L$  = number of external  $L$  lines. On the other hand, using the fact that both kinds of vertices involve three lines, we can write the total number of loops  $l$  as

$$l = \frac{1}{2}(V_a + V_b - E_L) + 1, \quad (24)$$

where we assumed that there are only external lines of  $L$ , since they are the only diagrams susceptible of comparison with the non-polynomial formulation. Combining both equations, we see that

$$f(G) = g^{\frac{3}{2}E_L - l - 1}. \quad (25)$$

Then, for any fixed  $E_L$ , increasing the number of loops increases the negative power of  $g$  (for diagrams with external lines of the other fields, the only difference is in the factor depending on the number of external lines). The same happens in the non-polynomial formulation, since the Lagrangian contains monomials with higher powers of the pion fields and derivatives, thus negative powers of the dimensionful coupling constant are required to keep the dimensionality of the Feynman amplitude constant.

We conclude then that the perturbation theory we are studying has the same perturbative parameter as the usual one, but starting with a different set of free fields.

### 3 The bosonic propagator to one-loop order.

As an example of an application we calculate here the one-loop correction to the propagators for the bosonic fields  $L$  and  $\theta$ . The full propagators will be constructed by using the free ones and the 1PI two-point functions for the vector fields. There are three of them, which (with an obvious notation) we denote by:  $\Gamma_{LL}$ ,  $\Gamma_{\theta\theta}$  and  $\Gamma_{L\theta}$ . Note that one cannot calculate the full  $\langle LL \rangle$  propagator, say, by knowing only the (one-loop) 1PI two-point function  $\Gamma_{LL}$  and summing the geometric series. Instead, the full  $\langle LL \rangle$  propagator will also receive contributions from the two-point functions  $\Gamma_{\theta\theta}$  and  $\Gamma_{L\theta}$ , because there is a mixed free propagator. This problem can be dealt with by working with the two-component field  $\Phi$  defined in subsection 2.1 above: one just calculates the full  $\Phi$  propagator to one-loop order, and then the full propagators of the original fields are read from the corresponding matrix elements. The full  $\Phi$  propagator  $G$  is given by

$$G = (D^{-1} - \Gamma)^{-1} \quad (26)$$

where  $D$  denotes the free  $\Phi$  propagator and  $\Gamma$  the 1PI two-point function of  $\Phi$  to one-loop order. Of course, the components of  $\Gamma$  are just  $\Gamma_{LL}$ ,  $\Gamma_{\theta\theta}$  and  $\Gamma_{L\theta}$ . To one-loop order they receive contribution from the Feynman diagrams shown in Fig.2. Naive power-counting gives the superficial degrees of divergence:

$$\omega[1] = 1, \omega[2] = 1, \omega[3] = 1, \omega[4] = 3, \omega[5] = 2. \quad (27)$$

We calculate them using dimensional regularization, obtaining the following contributions for each diagram

Diagram [1]:

$$I_{\mu\nu}^{ab}(p) = \frac{g}{24} \delta^{ab} p \left\{ \frac{1}{2} \left[ 3 \frac{p_\mu p_\nu}{p^2} - \delta_{\mu\nu} \right] + \left( \frac{\lambda - 1}{\lambda} \right) \left[ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \right\}. \quad (28)$$

Diagram [2]

$$I_{\mu\nu}^{ab}(p) = \frac{g}{2^4} \delta^{ab} p \left( \frac{p_\mu p_\nu}{p^2} + \delta_{\mu\nu} \right). \quad (29)$$

Diagram [3]

$$I_{\mu\nu}^{ab}(p) = -\frac{g}{2^4} \delta^{ab} p \left( \frac{p_\mu p_\nu}{p^2} + \delta_{\mu\nu} \right). \quad (30)$$

Diagram [4]

$$I_{\mu\nu}^{ab}(p) = \frac{-1}{2^6 g} \delta^{ab} p^3 \left( \frac{p_\mu p_\nu}{p^2} - \delta_{\mu\nu} \right). \quad (31)$$

Diagram [5]

$$I_{\mu\nu}^{ab}(p) = \frac{1}{2^5} \delta^{ab} p \epsilon_{\mu\alpha\nu} p_\alpha, \quad (32)$$

where  $p \equiv \sqrt{p^2}$ , and all combinatorial factors are already included. From these results one gets the matrix elements of  $\Gamma$  (see Fig.2):

$$\begin{aligned} \Gamma_{LL} &= \Gamma_{LL}^{[1]} + \Gamma_{LL}^{[2]} + \Gamma_{LL}^{[3]} \\ &= \frac{g}{2^5} p \left[ \left( \frac{\lambda-2}{\lambda} \right) \delta_{\mu\nu} + \left( \frac{\lambda+2}{\lambda} \right) \frac{p_\mu p_\nu}{p^2} \right] \\ \Gamma_{\theta\theta} &= \Gamma_{\theta\theta}^{[4]} \\ &= \frac{1}{2^6 g} p^3 (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \\ \Gamma_{L\theta} &= \Gamma_{\theta L} = \Gamma_{L\theta}^{[5]} \\ &= \frac{1}{2^5} p \epsilon_{\mu\nu\alpha} p_\alpha. \end{aligned} \quad (33)$$

None of the functions presented in (27) is divergent. Some explanation about the role of dimensional regularization is in order. It is well known [11] that dimensional regularization gives zero for the integrals  $\int d^D p / p^k$ ,  $k = 0, 1, 2, \dots$ , which are not defined for any integer  $D$ . These divergences correspond to infinities associated with self-contractions and are thus eliminated by normal-ordering, which we assume henceforward. We conclude that those are the only infinities we find at one loop order for these functions. More general diagrams are considered in section 4.

Substituting (33) into (26), and inverting the resulting matrix, one obtains the full propagators

$$G_{\mu\nu}^{LL} = \langle L_\mu L_\nu \rangle = a_1(p) \frac{p_\mu p_\nu}{p^2} + a_2(p) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (34)$$

$$G_{\mu\nu}^{\theta\theta} = \langle \theta_\mu \theta_\nu \rangle = b_1(p) \frac{p_\mu p_\nu}{p^2} + b_2(p) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (35)$$

$$G_{\mu\nu}^{L\theta} = \langle L_\mu \theta_\nu \rangle = c(p) \epsilon_{\mu\lambda\nu} \frac{p_\lambda}{p}, \quad (36)$$

where:

$$a_1(p) = \frac{1}{g^2(1 + \frac{p}{16g})} \quad (37)$$

$$a_2(p) = \frac{32\lambda(p/g)}{g^2(2048\lambda - 96\lambda(p/g) - 2(p/g)^2 + 3\lambda(p/g)^2)} \quad (38)$$

$$b_1(p) = \frac{1}{\lambda p^2} \quad (39)$$

$$b_2(p) = \frac{64(32\lambda - 2(p/g) + \lambda(p/g))}{p^2(2048\lambda - 96\lambda(p/g) - 2(p/g)^2 + 3\lambda(p/g)^2)} \quad (40)$$

$$c(p) = \frac{64\lambda((p/g) - 32)}{g^2(p/g)(2048\lambda - 96\lambda(p/g) - 2(p/g)^2 + 3\lambda(p/g)^2)}. \quad (41)$$

An important question to discuss at this point is the gauge-independence of the physical results that can be obtained from this one-loop calculation. To this end we just need to recall Slavnov's argument [4]: The relation between  $L_\mu$  and the pion field is

$$\begin{aligned} L_\mu(x) &= -\frac{1}{g} \partial_\mu \pi(x) - \frac{1}{g^{3/2}} \pi(x) \partial_\mu \pi(x) + \dots \\ \Rightarrow \partial \cdot L(x) &= -\frac{1}{g} \partial^2 \pi(x) + \dots, \end{aligned} \quad (42)$$

thus, when evaluating on-shell amplitudes of  $\pi$ , we get the relation

$$\lim_{p^2 \rightarrow 0} p^2 \langle \pi(p) \dots \rangle = \lim_{p^2 \rightarrow 0} \langle (-ip_\mu) L_\mu \dots \rangle. \quad (43)$$

The terms of higher order in  $\pi$  do not contribute to the amplitudes, because they are not one-particle connected.

Thus the physical amplitudes are completely determined by the correlation functions of  $\partial \cdot L$  in our case, the longitudinal part of the  $\langle LL \rangle$  propagator is the physically relevant one. From eqs. (34) and (37) we see that it is independent of  $\lambda$ .

## 4 Non-renormalization to one-loop order.

We show here that most of the one-loop diagrams are finite; in particular, all which contribute to the physical amplitudes: the ones with external lines of  $L$  only. We need some power-counting first. Note that the large momentum behaviour of the bosonic propagators can be summarized by saying that they go like  $k^{-r_\theta}$ , where  $r_\theta$  is the ‘number of  $\theta$ ’s in the propagator’; i.e.,  $\langle LL \rangle$  has  $r_\theta = 0$ ,  $\langle L\theta \rangle$  has  $r_\theta = 1$ , and  $\langle \theta\theta \rangle$  has  $r_\theta = 2$ . This provides an easy way to count the total power of  $k$  in a loop of bosonic fields: one just adds-up the  $r_\theta$ ’s of all the propagators involved. If the  $n$  external lines are  $L$ ’s, then one realises that the total number of  $\theta$ ’s in the propagators of the loop equals  $n$  (there is one  $\theta$  for each external line of  $L$ , since they must be connected to a vertex, which only has one  $\theta$ ). The superficial degree of divergence then equals  $3 - n$ , the 3 coming from the momentum integration. The functions with an odd number of lines of  $L$  vanish, and the two-point one was explicitly shown to be finite in the previous section. Thus the loops involving only bosonic fields propagators are finite.

On the other hand, the fermionic ghosts have a  $k^{-2}$  behaviour, as usual, and the vertex (with two ghosts and one  $L$ ) has one derivative. The only possibility of including ghosts in a one-loop diagram with external  $L$ ’s only is in a ghost loop. The superficial degree of divergence of such a diagram is  $\omega_n = 3 - 2n + n$ , where the 3 comes from the loop integration, the  $-2n$  from the ghost propagators, and the  $+n$  from the derivatives at the vertices. The only diagrams that might diverge are then the ones with  $n = 1, 2$ , or 3. The  $n = 1$  one vanishes trivially, and we have explicitly calculated the  $n = 2$  case, getting a finite result. There remains the ghost

triangle. Its superficial degree of divergence equals 0, implying that its divergent part (if any) is a constant independent of the external momenta, and may thus be obtained by putting all the external momenta equal to zero. We see immediately that this function is proportional to the integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu p^\lambda}{(p^2)^3}, \quad (44)$$

which vanishes. This completes the proof of non-divergence of the diagrams with external lines of  $L$  only.

The diagrams involving external lines of  $L$  are not the only ones one can show to be finite. Also the ones with external lines of  $\theta$  only are finite. The argument goes as follows: These diagrams can only have a loop containing  $\langle LL \rangle$  propagators. These propagators are longitudinal, thus in each vertex (connected to one of the external lines of  $\theta$ ) we have two propagators proportional to the corresponding momenta. Due to the presence of the Levi-Civita tensor, if the external  $\theta$  line is contracted with its momentum the diagram will be zero, since the sum of the three momenta flowing to the vertex is zero (i.e., the three momenta are in the same plane, and the vertex measures the volume they span). Of course the same happens with all the external lines. The situation becomes then similar to the one of  $QED$ , where one can prove that the proper vertex with external photon lines gives zero when one of the photon lines is contracted with its momentum. This reduces significantly the degree of divergence of those diagrams. A completely similar argument allows one to see that in our model the degree of divergence for a diagram with  $n$   $\theta$  lines becomes equal to  $3 - n$ , since we can factorize one momentum for each line. The case  $n = 0$  is trivial and for  $n = 2$  we have proved it gives a finite result. There remains only the case with three external lines of  $\theta$  which might diverge logarithmically. It is however finite because of the vanishing of the integral (44).



## 5 Conclusions.

We have seen that the (superficially divergent) 1-loop physical amplitudes derived from the polynomial Lagrangian are in fact convergent, calculating the bosonic propagator as an example. Although the issue of all-orders renormalizability remains an open question, this result already suggests that the number of counterterms required at each order may be substantially smaller than in the non-polynomial formulation. Of course the symmetries play a fundamental role in the imposition of constraints on the admissible counterterms. In particular, the *BRST* symmetry discussed in Appendix A, although softly broken by the term quadratic in  $L$ , might prove useful in that respect. The reason for that conjecture is that the model can be described in terms of the superfields  $\Psi$  and  $\bar{\mathcal{C}}$ , and thus some supersymmetry-like cancellation may occur. This requires the study of perturbation theory in terms of the corresponding supergraphs. Results about this approach will be reported elsewhere.

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## Appendix A: Path-Integral measure for $L$ .

As mentioned in the Introduction, the condition  $F_{\mu\nu} = 0$  is equivalent to  $L_\mu = \frac{1}{g^{1/2}} U \partial_\mu U^\dagger$ , and this implies the equivalence between the polynomial and non-polynomial theories. This equivalence, however, is only formal (‘classical’) until we justify the integration measure for  $L$ , which we have taken to be the trivial (flat) one. We now show that that measure is the proper one.

A non-trivial Jacobian factor in the measure for  $L$  may be expected since in principle one would write a delta-function imposing the pure-gauge condition as

$$\delta_{\text{pure gauge}}(L) = \delta[F(L)] J[L] , \quad (45)$$

where  $J[L] = \det \delta F / \delta L$ . Let us show that  $J[L]$  is just a field-independent constant. A simple calculation shows that  $J[L]$  can be written as a functional integral over some (new) vector ghosts  $\bar{c}_\mu, c_\mu$

$$\begin{aligned} J[L] &= \int \mathcal{D}\bar{c}_\mu \mathcal{D}c_\mu e^{\int d^3x \mathcal{M}} \\ \mathcal{M} &= \bar{c}_\mu \epsilon^{\mu\nu\lambda} D_\nu c_\lambda . \end{aligned} \quad (46)$$

Due to the presence of the delta-function of  $F(L)$  in (45), we see that the  $L$  appearing in the covariant derivative in (46) is a pure gauge. Then we can write  $\mathcal{M}$  as

$$\mathcal{M}(x) = \bar{c}_\mu(x) U^\dagger(x) \epsilon^{\mu\nu\lambda} \partial_\nu [U(x) c_\lambda] , \quad (47)$$

and the factors  $U(x), U^\dagger(x)$  can be eliminated by a unitary transformation of the ghosts

$$c'_\mu(x) = U(x) c_\mu(x) , \quad \bar{c}'_\mu(x) = \bar{c}_\mu(x) U^\dagger(x) . \quad (48)$$

Thus we have seen that the ghost Lagrangian  $\mathcal{M}$  can be safely disregarded. It is interesting to note, however, that had we kept this factor, a new *BRST* symmetry would have emerged, as we show in what follows<sup>3</sup>. The  $\delta_{\text{pure gauge}}$  of eq.(45) can

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<sup>3</sup>We follow the conventions of ref. [11].

be written as a functional integral over the Lagrange multiplier  $\theta_\mu$  and the vector ghosts:

$$\delta_{\text{puregauge}}(L) = \int \mathcal{D}\theta_\mu \mathcal{D}\bar{c}_\mu \mathcal{D}c_\mu \exp[iS_{\text{BRST}}] , \quad (49)$$

where

$$S_{\text{BRST}} = \int d^3x [g\theta_\mu \cdot F^\mu(L) - i\epsilon^{\mu\nu\lambda}\bar{c}_\mu \cdot D_\nu c_\lambda] , \quad (50)$$

where  $F^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$ . There appears then the symmetry

$$\begin{aligned} \delta L^\mu &= i\bar{\epsilon}c^\mu \quad , \quad \delta \bar{c}^\mu = \bar{\epsilon}\theta^\mu \\ \delta c^\mu &= 0 \quad , \quad \delta \theta^\mu = 0 , \end{aligned} \quad (51)$$

where  $\bar{\epsilon}$  is a real fermionic constant (the ghost fields are also assumed to be real). The symmetry transformations (51) can be understood also as translation invariance in a new Grassmannian coordinate, defining the ‘superfields’

$$\begin{aligned} \Psi_\mu(x, \bar{\xi}) &= L_\mu + i\bar{\xi}\bar{c}_\mu \\ \bar{\mathcal{C}}_\mu(x, \bar{\xi}) &= \bar{c}_\mu + \bar{\xi}\theta_\mu , \end{aligned} \quad (52)$$

and noting that  $S_{\text{BRST}}$  can then be expressed solely in terms of  $\Psi$  and  $\bar{\mathcal{C}}$ :

$$S_{\text{BRST}} = \int d^3x d\bar{\xi} \bar{\mathcal{C}}_\mu \cdot F^\mu(\Psi) . \quad (53)$$

The form of eq.(53) resembles a delta-function constraining  $F(\Psi)$  to be zero. It must be noted, however, that when all the superfields are expanded in components, one recovers an action which already includes the ghosts. To complete the full (yet without gauge-fixing) polynomial Lagrangian we have to include in (53) the  $L^2$  term. This can be written also in terms of the superfields, but the translation invariance in  $\bar{\xi}$  is lost:

$$S_{\text{inv.}} = \int d^3x d\bar{\xi} \left[ \frac{1}{2}g^2\bar{\xi}\bar{\Psi}_\mu \cdot \Psi^\mu - \bar{\mathcal{C}}_\mu \cdot F^\mu(\Psi) \right] . \quad (54)$$

## Appendix B: Ghost Lagrangian and Gribov ambiguities.

We discuss here some properties related to the ghost Lagrangian  $\mathcal{L}_{gh}$ , corresponding to the covariant gauge-fixing term  $\mathcal{L}_{g.f.}$ . Despite being formally identical to the one of Yang-Mills theory in  $2 + 1$  dimensions (with  $L_\mu$  as the gauge field, and the gauge condition affecting  $L$  instead of  $\theta$ ),  $\mathcal{L}_{gh}$  has a very different meaning in the non-linear  $\sigma$ -model. Even though  $\mathcal{L}_{gh}$  looks ‘non-Abelian’ because of the covariant derivatives, the gauge group is Abelian. A consequence of this is that the functional integral over the ghost fields exactly reproduces the ‘Faddeev-Popov functional’  $\Delta_{F-P}$ , the gauge-invariant object defined by the equation:

$$1 = \Delta_{F-P}(L) \int \mathcal{D}\omega \, \delta[\partial \cdot \theta^\omega - f] \, , \quad \theta^\omega = \theta + D\omega \, , \quad (55)$$

(and a Gaussian average over  $f$  is performed afterwards, as usual). This is not what happens in Yang-Mills theory. There, the integration over the ghosts yields a functional which coincides with  $\Delta_{F-P}$  only on the gauge-fixed configurations, and it is consequently non gauge-invariant<sup>4</sup>.

There is also a difference regarding the existence of zero-modes for the  $F - P$  operator  $\partial \cdot D$  and the Gribov ambiguities [12]. If we want to look for gauge-equivalent configurations of the fields which satisfy the same gauge-fixing condition imposed in (55), we have to study the existence of non-zero  $\omega$ ’s satisfying

$$\partial \cdot \theta^\omega - f = \partial \cdot \theta - f \, . \quad (56)$$

This is equivalent to

$$\partial D\omega = 0 \, , \quad (57)$$

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<sup>4</sup>The situation for the polynomial formulation of the non-linear  $\sigma$ -model is similar to the one of  $QED$ , where one can rewrite the Faddeev-Popov functional as an integral over the ghosts, and the result is (trivially) gauge-invariant.

which is the same zero-mode equation one gets in Yang-Mills theory. However, we do not have any gauge-fixing condition on  $L$ , but rather the constraint  $F_{\mu\nu}(L) = 0$ . There are well-known types of zero-modes satisfying this constraint: they are given by configurations with half-integer topological charge, and are fermionic in character [13].

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## Figure Captions.

**Figure 1** Feynman rules in the polynomial formulation. The full line represents the  $L_\mu$  propagator, the wavy line the  $\theta_\mu$  propagator, and the mixed line the  $L_\mu - \theta_\mu$  propagator. The dashed line corresponds to the ghost.

**Figure 2** One-loop contribution to bosonic two-point functions.